TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 261, Number 2, October 1980

OCTONION PLANES OVER LOCAL RINGS[1]

BY

ROBERT BIX

ABSTRACT. Let $\mathfrak{O}$ be an octonion algebra which is a free module over a local ring R and let $J = H(\mathfrak{O}_3, \gamma)$ be the quadratic Jordan algebra of Hermitian 3-by-3 matrices over R. We define the octonion plane determined by J and prove that every collineation is induced by a norm semisimilarity of J. We classify the subgroups of the collineation group normalized by the little projective group.

In [5, p. 49] Faulkner proved that the little projective group of an octonion plane over a field is simple. This paper generalizes his result to octonion planes over local rings and extends it in the field case. We classify those subgroups of the collineation group of an octonion plane over a local ring which are normalized by the little projective group. This parallels the results of Klingenberg and Bass classifying those subgroups of the general linear group over a local ring which are normalized by the special linear group [11, p. 84].

Specifically, let (R, m) be a local ring and let $\mathfrak{O}$ be an octonion R-algebra which is a free R-module. Let $J = H(\mathfrak{O}_3, \gamma)$ be the quadratic Jordan algebra of Hermitian 3×3 matrices over $\mathfrak{O}$ and let N be the generic norm on J. Let Γ be the group of semilinear R-module automorphisms (ϕ, σ) of J such that there is $\rho \in R - m$ with $N(\phi x) = \rho N(x)^\sigma$ for $x \in J \otimes R[\eta]$, $R[\eta]$ a polynomial ring. Let $G = \{\phi \in \Gamma | \sigma = 1\}$ and let $S = \{\phi \in G | \rho = 1\}$. If H is a subgroup of Γ and I is an ideal of R, let $H_I = \{\phi \in H | \phi(x) \equiv x \pmod{IJ},\ x \in J\}$. Our main theorem states that a subgroup N of Γ is normalized by S if and only if $S_I \subseteq N \subseteq (R - m)\Gamma_I$ for an ideal I of R. Since the collineation group of the octonion plane determined by J is isomorphic to $\Gamma/(R - m)$, this theorem classifies the subgroups of the collineation group normalized by the little projective group, the image of S in the collineation group.

§1 establishes notation and preliminary results. For an ideal I of R, let T_I be the subgroup of S generated by

$$\left\{ T_{x,y} | x \in IJ, y \in J - mJ, y^\# = 0, T(x, y) = 0 \right\},$$

where $y^\#$ is the "adjoint" of y, $T(x, y)$ is the generic trace form, and $T_{x,y}$ is the algebraic transvection $1 + V_{x,y} + U_x U_y$. We develop the geometry of the octonion plane in §2 and use it in §3 to prove that a subgroup of Γ which is normalized by S

Received by the editors June 14, 1979.

AMS (MOS) subject classifications (1970). Primary 17C40; Secondary 20H25.

Key words and phrases. Octonion planes, exceptional Jordan algebras, norm semisimilarities, orthogonal groups over local rings.

[1]The contents of this paper were presented to the Conference on Jordan Algebras, Mathematisches Forschungsinstitut Oberwolfach, August 22, 1979.

0002-9947/80/0000-0453/$06.50

and not contained in $R - m$ contains T_I for a nonzero ideal I of R. §4 describes generators of the "congruence subgroups" of the orthogonal group of a nondegenerate quadratic form over a local ring containing a hyperbolic plane. We use this in §5 to construct the elements of S_I that fix a matrix unit e_1 and $J_0(e_1)$, from which we prove in §6 that $S_I = T_I$ for every ideal I of R. In §7 we prove our main theorem, which follows directly from the results of §§3 and 6. As corollaries, we construct the normal subgroups of S and G. In §8, we prove that every collineation of two octonion planes is induced by a norm semisimilarity. In particular, the collineation group of an octonion plane is isomorphic to $\Gamma/(R - m)$, as noted in the preceding paragraph.

The objects studied and the results sought in this paper are based on the work of Faulkner on octonion planes [5]. This work was based in turn on the results of Springer, Veldkamp, and Jacobson, cited in the bibliography of [5].

1. Preliminaries. In this section we establish notation and basic results. We show that we need only consider invertible elements to establish identities for algebras defined by the Freudenthal-Springer-Tits construction over commutative rings [10]. We prove this by using localization at generic elements to replace Zariski topology arguments, as in [1, Chapter III]. We apply the reduction to invertible elements to derive the basic properties of algebraic transvections $T_{x,y}$ in such algebras. Finally we present several basic results on octonion algebras over commutative rings.

All commutative rings have 1 and all modules and algebras are unital. Let R and R' be commutative rings.

Let M and M' be R-modules. A map Q: $M \to M'$ is called quadratic if $Q(\alpha a) = \alpha^2 Q(a)$ for $\alpha \in R$ and $a \in M$ and if $Q(a + b) - Q(a) - Q(b)$ is bilinear in $a, b \in M$. Write $Q(a + b) - Q(a) - Q(b)$ as $\partial_{a,b}Q$ or $Q(a, b)$. If $M' = R$, we call Q a quadratic form and $Q(a, b)$ the associated bilinear form. A cubic form $(N, \partial N)$ is a map N: $M \to R$ and a map ∂N: $M \times M \to R$ such that $\partial_a N|_b \equiv \partial N(a, b)$ is linear in a and quadratic in b, $N(\alpha a) = \alpha^3 N(a)$, $\partial_a N|_a = 3N(a)$, and $N(a + b) = N(a) + \partial_a N|_b + \partial_b N|_a + N(b)$ for $\alpha \in R$ and $a, b \in M$.

DEFINITION 1.1. A *cubic algebra* is an R-module J, an element $1 \in J$, a quadratic map $a \to a^\#$ from J to itself, and a cubic form $(N, \partial N)$ on J such that:

(1) $x^{\#\#} = N(x)x$,

(2) $N(1) = 1$,

(3) $T(x^\#, y) = \partial_y N|_x$, where $T(x, y) = (\partial_x N|_1)(\partial_y N|_1) - \partial_{x,y}(\partial_1 N|)$,

(4) $1^\# = 1$,

(5) $1 \times y = T(y)1 - y$, where $x \times y = (x + y)^\# - x^\# - y^\#$ and $T(y) = T(y, 1)$, and (1)–(5) hold under all scalar extensons of R [10, p. 495]. □

A cubic algebra J is a quadratic Jordan algebra under $U_x y = T(x, y)x - x^\# \times y$. Every $x \in J$ satisfies $x^3 - T(x)x^2 + T(x^\#)x - N(x)1 = 0$ [10, p. 499]. We note that

$$T(a_1 \times a_2, a_3) = \partial_{a_1,a_2}\partial_{a_3}N| = N\Big(\sum a_i\Big) - \sum N(a_i) - \sum_{i \neq j} \partial_{a_i}N|_{a_j}$$

is symmetrical in the a_i.

If M and M' are modules over R and R' respectively, a semilinear homomorphism $\phi\colon M \to M'$ is a homomorphism of the additive groups such that there is a ring isomorphism $\sigma\colon R \to R'$ with $\phi(\alpha a) = \alpha^{\sigma}\phi(a)$ for $\alpha \in R$, $a \in M$.

DEFINITION 1.2. If J and J' are cubic algebras over R and R' respectively, a *norm semisimilarity* $\phi\colon J \to J'$ is a semilinear isomorphism (ϕ, σ) such that there is a unit ρ of R' with $N'(\phi x) = \rho N(x)^{\sigma}$ and $\partial_{\phi(x)}N'|_{\phi(y)} = \rho(\partial_x N|_y)^{\sigma}$ for $x, y \in J$. Equivalently, $N'(\phi x) = \rho N(x)^{\sigma}$ for $x \in J \otimes_R R[\eta]$, where $R[\eta]$ is a polynomial ring over R and we extend (ϕ, σ) by $\eta^{\sigma} = \eta$. Let $\Gamma = \Gamma(J)$ be the group of norm semisimilarities from J to itself. Call $G = G(J) = \{\phi \in \Gamma | \sigma = 1\}$ the group of norm similarities and $S = S(J) = \{\phi \in G | \rho = 1'\}$ the group of norm preserving transformations. □

LEMMA 1.3. *Let M be an R-module and let $R[\eta_i]$ be a polynomial ring. Let $0 \neq g \in M \otimes_R R[\eta_i]$ and let $f(\eta_i) \in R[\eta_i]$ satisfy $f(\alpha_i) = 1$ for some $\alpha_i \in R$. Then $fg \neq 0$.*

PROOF. Let $N \neq 0$ be the submodule of M spanned by the elements needed to write g, and consider $g \in N \otimes R[\eta_i]$. Since N is finitely spanned, there is a maximal ideal m of R such that $N/mN \neq 0$ [**4**, p. 7]. Let g' be the image of g in $N/mN \otimes_{R/m} R/m[\eta_i]$ and f' the image of f in $R/m[\eta_i]$. $g' \neq 0$, since the coefficients of g' span N/mN, and $f' \neq 0$, since $f(\alpha_i) = 1$. Then $f'g' \neq 0$, since R/m is a field, so $fg \neq 0$ in $N \otimes R[\eta_i] \subseteq M \otimes R[\eta_i]$. □

Let J be a cubic algebra. $x \in J$ is invertible if and only if $N(x)$ is a unit, and then $x^{-1} = N(x)^{-1}x^{\#}$ [**10**, p. 500].

PROPOSITION 1.4. *Let J be a cubic R-algebra. Let $F(x_i) = 0$ be an identity which holds for all invertible x_i in every scalar extension of J. Then it holds for all x_i in J.*

PROOF. Let $d_1, \ldots, d_n \in J$ and let $\{\xi, \eta_1, \ldots, \eta_n\}$ be indeterminates. Set $y_i = \xi 1 + \eta_i d_i \in J \otimes R[\xi, \eta_i]$. Let $f = N(y_1) \cdots N(y_n) \in R[\xi, \eta_i]$ and let $R[\xi, \eta_i]_f$ be the localization of $R[\xi, \eta_i]$ at the powers of f. Since $N(y_i)$ is a unit in $R[\xi, \eta_i]_f$, y_i is invertible in $J \otimes R[\xi, \eta_i]_f$. Then $F(y_i) = 0$ in $J \otimes R[\xi, \eta_i]_f$, so $f^t F(y_i) = 0$ in $J \otimes R[\xi, \eta_i]$ for some positive integer t. Since $f(1, 0, \ldots, 0) = 1$, $F(y_i) = 0$ [Lemma 1.3]. Setting $\xi = 0$ and $\eta_i = 1$ gives $F(d_i) = 0$. □

The next corollary partially answers a question of McCrimmon [**10**, p. 501].

COROLLARY 1.5. *If J is a cubic R-algebra, $[N(U_x y) - N(x)^2 N(y)]J = 0$ and $[N(x^{\#}) - N(x)^2]J = 0$ for $x, y \in J$. Thus $N(U_x y) = N(x)^2 N(y)$ and $N(x^{\#}) = N(x)^2$ if J is a faithful R-module.*

PROOF. By Proposition 1.4, we can assume that x and y are invertible. $N(U_x y)U_x y = N(x)^2 N(y) U_x y$ [**10**, p. 499] and $N(x^{\#})x^{\#} = (x^{\#})^{\#\#} = (x^{\#\#})^{\#} = N(x)^2 x^{\#}$ [Definition 1.1]. Since $U_x y$ and $x^{\#}$ are invertible, it suffices to prove that $\alpha z = 0$ implies $\alpha J = 0$ for $\alpha \in R$ and invertible $z \in J$. $\alpha z = 0$ gives

$$\alpha N(z)1 = \alpha z^3 - \alpha T(z)z^2 + \alpha T(z^{\#})z = U_z \alpha z - T(\alpha z)z^2 + T(z^{\#})\alpha z = 0.$$

Since $N(z)$ is a unit, $\alpha 1 = 0$. For $w \in J$,

$$\alpha w = \alpha\big[T(w)1 - 1 \times w\big] = T(w)(\alpha 1) - (\alpha 1) \times w = 0. \quad \square \qquad \text{(i)}$$

DEFINITION 1.6. If M is a finitely spanned, projective R-module, a symmetric bilinear form $B(x, y)$ on M is called nondegenerate if $M \cong \mathrm{Hom}_R(M, R)$ via $x \to B(x, -)$. This is equivalent to the condition that B induces a nondegenerate form on M/mM over R/m for every maximal ideal m of R, by [**11**, pp. 141–144] and localization. A quadratic form Q on M is called nondegenerate if the associated bilinear form $Q(x, y)$ is nondegenerate. □

Let J be a cubic R-algebra. Assume that J is finitely spanned R-projective and that $T(x, y)$ is nondegenerate. If ϕ: $J \to J$ is σ-semilinear, define a σ^{-1}-semilinear homomorphism ϕ^*: $J \to J$ by $T(\phi^* x, y) = T(x, \phi y)^{\sigma^{-1}}$. If ϕ is a semilinear isomorphism, so is ϕ^*, and $\phi^{*^{-1}} = \phi^{-1*}$.

LEMMA 1.7. *Let J be a faithful cubic algebra such that $T(x, y)$ is nondegenerate. If $(\phi, \rho, \sigma) \in \Gamma(J)$, then $\phi^* \in \Gamma(J)$, $(\phi x)^\# = \rho\phi^{*^{-1}}(x^\#)$, $\phi U_x \phi^* = U_{\phi x}$, and $\phi V_{x,y}\phi^{-1} = V_{\phi x, \phi^{*^{-1}} y}$ for $x, y \in J$.*

PROOF. Extend ϕ and ϕ^* to $J \otimes R[\xi, \eta]$. Let $y = \xi 1 + \eta x$, $x \in J$. $T((\phi y)^\#, \phi z) = \rho T(y^\#, z)^\sigma$ for $z \in J \otimes R[\xi, \eta]$, so $(\phi y)^\# = \rho\phi^{*^{-1}}(y^\#)$. Taking norms gives $N(y)^{2\sigma} = \rho N(\phi^{*^{-1}} y^\#)$, by Corollary 1.5. Replacing y by $y^\#$ gives $N(y)^{4\sigma} = \rho N(y)^{3\sigma} N(\phi^{*^{-1}} y)$. By Lemma 1.3, $N(y)^\sigma = \rho N(\phi^{*^{-1}} y)$. Specializing $\xi = 0$ and $\eta = 1$ commutes with $\#$, ϕ, and ϕ^* to give $(\phi x)^\# = \rho\phi^{*^{-1}}(x^\#)$ and $N(x)^\sigma = \rho N(\phi^{*^{-1}} x)$, $x \in J$. Then $\phi^* \in \Gamma(J)$ and replacing ϕ by $\phi^{*^{-1}}$ above gives $\phi(x^\#) = \rho(\phi^{*^{-1}} x)^\#$. The last two statements of the lemma now follow directly from the definitions of U_x and $V_{x,y}$ [**5**, p. 11]. □

If J is a cubic algebra and $x, y \in J$, set $T_{x,y} = 1_J + V_{x,y} + U_x U_y$ where 1_J is the identity map on J. $T_{\alpha x, y} = T_{x, \alpha y}$ for $\alpha \in R$.

LEMMA 1.8. *Let J be a faithful cubic R-algebra such that $T(x, y)$ is nondegenerate, and let $w, x, y, z \in J$.*

(1) $T_{x,y} T_{x,-y} = T_{-x, U_y x} = T_{U_x y, -y}$.

(2) *If x and y are invertible,* $T_{x,y} = U_{x+y^{-1}} U_y = U_x U_{x^{-1}+y}$.

(3) $\phi T_{x,y} \phi^{-1} = T_{\phi x, \phi^{*^{-1}} y}$ *for* $\phi \in \Gamma$.

(4) $T_{x,y}^* = T_{y,x}$.

(5) $[N(T_{x,y} z) - (1 + N(x)N(y) + T(x, y) + T(x^\#, y^\#))^2 N(z)]J = 0$.

(6) *If $T(x, y) = 0$ and either $x^\# = 0$ or $y^\# = 0$, then $T_{x,y}^{-1} = T_{-x,y}$ and $T_{x,y} \in S$.*

PROOF. (1) follows from the identities QJ3, QJ27, and QJ28 [**5**, pp. 6–7]. (2) $U_y U_{x, y^{-1}} U_y = U_{U_y x, y} = U_y V_{x,y}$ [QJ2, QJ3]. Then $U_{x, y^{-1}} U_y = V_{x,y}$ and

$$U_{x+y^{-1}} U_y = U_x U_y + U_{x, y^{-1}} U_y + U_{y^{-1}} U_y = T_{x,y}.$$

The other equality is proved similarly. (3) follows from Lemma 1.7. (4) holds, since $T(U_x z, w) = T(z, U_x w)$ and $T(V_{x,y} z, w) = T(z, V_{y,x} w)$ follow from the definitions of U_x and $V_{x,y}$ and the symmetry of $T(a_1 \times a_2, a_3)$. (5) We can assume that x and y are invertible by Proposition 1.4. Then by (2) and Corollary 1.5, for $w \in J$,

$$\begin{aligned} N(T_{x,y} z) w &= N(U_x U_{x^{-1}+y} z) w = N(x)^2 N(x^{-1} + y)^2 N(z) w \\ &= N(x)^2 \left[N(x^{-1}) + T\big((x^{-1})^\#, y\big) + T(x^{-1}, y^\#) + N(y) \right]^2 N(z) w \\ &= \left[1 + T(x, y) + T(x^\#, y^\#) + N(x)N(y) \right]^2 N(z) w, \end{aligned}$$

since $x^{-1} = N(x)^{-1}x^{\#}$. (6) Assume that $x^{\#} = 0$; the case $y^{\#} = 0$ is similar. $U_x y = T(x, y)x - x^{\#} \times y = 0$, so (1) shows that $T_{x,y}^{-1} = T_{-x,y}$. $N(x)1 = 0$, since $x^{\#} = 0$ and $x^{\#} \times (x \times 1) = N(x)1 + T(x^{\#}, 1)x$ is a linearization of $x^{\#\#} = N(x)x$. Taking $\alpha = N(x)$ in equation (i) shows that $N(x)J = 0$. Then $T_{x,y} \in S$, by (5) and the faithfulness of J. □

$T_{x,y}$ is called an *algebraic transvection* if the conditions of Lemma 1.8(6) are satisfied.

Let M be a finitely generated, projective R-module. If R is local, M is free. If R is any commutative ring, M is said to have rank n if $M \otimes_R R_p$ is a free R_p-module of rank n for every prime p of R, where R_p is the localization of R at p [**4**, pp. 24, 27].

A composition algebra (D, d) over R is a unital alternative R-algebra D with involution d such that D is finitely spanned R-projective and $xx^d = n(x)1 = x^d x$ for $x \in J$, where $n(x)$ is a nondegenerate quadratic form on D [Definition 1.6]. Set $t(x) = n(x, 1)$, so $t(x)1 = x + x^d$. We define a quaternion algebra as a composition algebra of rank 4 and an octonion algebra as a composition algebra of rank 8.

Let $(\mathfrak{O}, d)$ be an octonion R-algebra. Let $\mathfrak{O}_3$ be the nonassociative algebra of 3-by-3 matrices over $\mathfrak{O}$, and let $e_i, e_{ij} \in \mathfrak{O}_3$ be the canonical matrix units. Let $\gamma_1, \gamma_2, \gamma_3$ be units of R and set $\gamma = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3$. Let

$$H(\mathfrak{O}_3, \gamma) = \Big\{ \sum \alpha_i e_i + \sum a_i[jk] \,|\, \alpha_i \in R, a_i \in \mathfrak{O} \Big\},$$

where (ijk) is a cyclic permutation of (123) and

$$a[jk] = \gamma_k a e_{jk} + \gamma_j a^d e_{kj}.$$

For $x = \sum \alpha_i e_i + \sum a_i[jk]$ and $y = \sum \beta_i e_i + \sum b_i[jk]$, set

$$N(x) = \alpha_1\alpha_2\alpha_3 - \sum \alpha_i\gamma_j\gamma_k n(a_i) + \gamma_1\gamma_2\gamma_3 t((a_1a_2)a_3),$$

$$T(x, y) = \sum \alpha_i\beta_i + \sum \gamma_j\gamma_k n(a_i, b_i),$$

$$x^{\#} = \sum (\alpha_j\alpha_k - \gamma_j\gamma_k n(a_i))e_i + \sum \big(\gamma_i(a_ja_k)^d - \alpha_i a_i\big)[jk],$$

and

$$1 = e_1 + e_2 + e_3.$$

Then $J = H(\mathfrak{O}_3, \gamma)$ is a cubic algebra [**10**, p. 503]. $T(x, y)$ is nondegenerate, since it induces a nondegenerate form on J/mJ for every maximal ideal m of R. If $x, y \in J$ have coefficients in a subalgebra of $\mathfrak{O}$ generated by a single element, $U_x z = (xz)x = x(zx) \equiv xzx$ and

$$T_{x,y}z = (1 + xy)z(1 + yx) \tag{ii}$$

for $z \in J$ [**5**, p. 16]. As in Lemma 1.8(7), $e_j^{\#} = 0$ and

$$\begin{aligned} \{ T_{x,e_j} \,|\, T(x, e_j) = 0, x \in J \} &= \{ T_{x,e_j} \,|\, x \in J_0(e_j) + J_{1/2}(e_j) \} \\ &= \{ T_{p[ij]+q[jk],e_j} \,|\, p, q \in \mathfrak{O} \} \end{aligned} \tag{iii}$$

by the Peirce relations [**5**, p. 15]. Write $T = T_{p[ji],e_j}$, $p \in \mathfrak{O}$. Then

$$T(x) = \big(1 + \gamma_j p^d e_{ij}\big)x\big(1 + \gamma_i p e_{ji}\big)$$

for $x \in J$. This gives

$$T(e_i) = e_i, \quad T(e_j) = e_j + p[ji] + \gamma_i\gamma_j n(p)e_i,$$
$$T(e_k) = e_k, \quad T(q[kj]) = q[kj] + \gamma_j qp[ki],$$
$$T(q[ki]) = q[ki], \quad T(q[ji]) = q[ji] + \gamma_i\gamma_j n(p, q)e_i.$$

Lastly we present several basic properties of octonion algebras over commutative rings. As noted earlier, a finitely spanned, projective module M over a local ring (R, m) is free. In fact, if $x_1, \ldots, x_n \in M$ induce a vector space basis of M/mM over R/m, $x_1, \ldots, x_n$ are a free basis for M over R [**4**, p. 24]. In particular, a composition algebra over a local ring has this property.

LEMMA 1.9. *Let (C, d) be a composition algebra of rank greater than one over a local ring (R, m). Then there are subalgebras $R = R1 \subset C_1 \subset C_2 \subset \cdots \subset C_t = C$ such that C_i is a composition algebra of rank 2^i and $C_{i+1} = C_i \oplus C_i p_{i+1}$ for some $p_{i+1} \in C_{i+1}$, where $n(p_{i+1}) = -v_{i+1} \in R - m$ and the elements of C_{i+1} multiply by*

$$(a + bp_{i+1})(c + ep_{i+1}) = (ac + v_{i+1}e^d b) + (ea + bc^d)p_{i+1},$$

for $a, b, c, e \in C_i$.

PROOF. $1 \in C - mC$, so $R = R1 \subset C$. Take $a' \in C/mC - (R/m)1$ such that n is nondegenerate on $(R/m)1 + (R/m)a'$. Let $C_1 = R1 + Ra$, where a is a preimage of a'. C_1 is a subalgebra of C, since $a^2 - t(a)a + n(a)1 = 0$. By induction, assume that we have found $C_s \neq C$. Since n is nondegenerate on C and C_s, $C = C_s \oplus C_s^{\perp}$ and n is nondegenerate on $C_s^{\perp}$. Take $p_{s+1} \in C_s^{\perp}$ such that $n(p_{s+1}) \in R - m$, and set $C_{s+1} = C_s \oplus C_s p_{s+1}$. The lemma follows as in [**7**, pp. 163–164]. □

LEMMA 1.10. *Let Q be a quaternion algebra over a commutative ring R.*
(1) *Q is a central separable associative R-algebra.*
(2) *Q is generated as an algebra without 1 by $\{ab - ba | a, b \in Q\}$.*
(3) *If (R, m) is local, there is $c \in Q$ such that $c - c^d$ is invertible.*

PROOF. (1) Q is associative and $R = R1 \subset Q$, by Lemma 1.9 and localization. For every maximal ideal m of R, Q/mQ is R/m-central simple. It follows that Q is R-central separable, by the associative analogue of [**2**, Theorem 1.8]. (2) By Nakayama's Lemma [**4**, p. 7], it suffices to establish (2) modulo every maximal ideal of R, so we can assume that $R = F$ is a field. If E is the algebraic closure of F, $Q \otimes E$ is the algebra of 2-by-2 matrices over E and the result follows. (3) If char $R/m \neq 2$, choose $c \in 1^{\perp}$ such that $n(c)$ is a unit; then $c^d = -c$ and $c - c^d = 2c$ is invertible. If char $R/m = 2$, choose $c \in Q$ such that $c + c^d = n(1, c)1 \in (R - m)1$. □

Let $(\mathfrak{O}, d)$ be an octonion R-algebra. Set $[a, b, c] = (ab)c - a(bc)$ and $[a, b] = ab - ba$ for $a, b, c \in \mathfrak{O}$. Let the nucleus of $\mathfrak{O}$ be $\{a \in \mathfrak{O} | [x, a, y] = 0, x, y \in \mathfrak{O}\}$.

LEMMA 1.11. *R is the nucleus of $\mathfrak{O}$ and $R = \{a \in \mathfrak{O} | [a, x] = 0, x \in \mathfrak{O}\}$.*

PROOF. It suffices to show this for every localization of R, so we can assume that (R, m) is local. By Lemma 1.9, take a quaternion subalgebra Q of $\mathfrak{O}$ and $p \in \mathfrak{O}$ such that $\mathfrak{O} = Q \oplus Qp$ and $p^2 = \nu \in R - m$. Let $a + bp$ be in the nucleus, $a, b \in Q$. For any $c, e \in Q$, $0 = [c, a + bp, e] = (b[c, e^d])p$, so $b = 0$ [Lemma 1.10(2)]. For any $c \in Q$, $0 = [cp, a, p] = \nu[c, a^d]$, so $a \in R$ [Lemma 1.10(1)] and R is the nucleus of $\mathfrak{O}$. Next, suppose that $a + bp$ satisfies $[a + bp, x] = 0$, $x \in \mathfrak{O}$. If $c \in Q$, $0 = [c, a + bp] = [c, a] + b(c - c^d)p$; so $a \in R$ and $b = 0$ [Lemma 1.10]. □

LEMMA 1.12. *The right ideals of* $\mathfrak{O}$ *are* $I\mathfrak{O}$, I *an ideal of* R.

PROOF. Let C be the subalgebra of $\mathrm{End}_R(\mathfrak{O})$ generated by the right multiplications by elements of $\mathfrak{O}$. Suppose that R is a field. $\mathfrak{O}$ is an irreducible C-module, since it has no nontrivial right ideals [7, p. 170]. The commutant of C is the set of left multiplications by elements of the nucleus R, so the density theorem gives $C = \mathrm{End}_R(\mathfrak{O})$ [6, p. 41]. Now let R be arbitrary. As above, for every maximal ideal m of R, the image of C in $\mathrm{End}_R(\mathfrak{O})/m\,\mathrm{End}_R(\mathfrak{O}) \cong \mathrm{End}_{R/m}(\mathfrak{O}/m\mathfrak{O})$ is the entire algebra. $\mathrm{End}_R(\mathfrak{O})$ is finitely spanned, since $\mathfrak{O}$ is finitely spanned R-projective [4, p. 18]. Then $C = \mathrm{End}_R(\mathfrak{O})$, by Nakayama's Lemma. Since $\mathfrak{O}$ is finitely spanned R-projective, there are $\phi_1, \ldots, \phi_n \in \mathrm{Hom}_R(\mathfrak{O}, R)$ and $a_1, \ldots, a_n \in \mathfrak{O}$ such that $x = \Sigma\phi_i(x)a_i$ for $x \in \mathfrak{O}$ [4, p. 4]. Since $\mathfrak{O}$ is projective of rank 8, $\mathfrak{O}$ is R-faithful and we can identify $R \subset \mathfrak{O}$ and $\mathrm{Hom}_R(\mathfrak{O}, R) \subset \mathrm{End}_R(\mathfrak{O})$. Since $C = \mathrm{End}_R(\mathfrak{O})$, there are $\tau_i \in C$ such that τ_i induces ϕ_i. If $N \subset \mathfrak{O}$ is a right ideal, $\tau_i(N) \subset N \cap R$. Then the equation $x = \Sigma\tau_i(x)a_i$ yields $N = (N \cap R)\mathfrak{O}$. □

2. Geometry of octonion planes. Henceforth let (R, m) be a local ring, $\mathfrak{O}$ an octonion algebra, and $J = H(\mathfrak{O}_3, \gamma)$. In this section we develop the geometry of the octonion plane for use in §3.

Let $\Pi = \{x \in J - mJ \mid x^\# = 0\}$. For $x \in \Pi$, let x_* and x^* be two copies of Rx. The octonion plane PJ consists of points x_* and lines x^*, $x \in \Pi$, with defining relations [5, Chapter III]:

$$x_* | y^*, \; x_* \text{ is on } y^*, \text{ if } V_{x,y} = 0,$$

$$x_* \sim y^*, \; x_* \text{ is connected to } y^*, \text{ if } T(x, y) \in m,$$

$$x_* \sim y_*, \; x_* \text{ is connected to } y_*, \text{ if } x \times y \in mJ,$$

$$x^* \sim y^*, \; x^* \text{ is connected to } y^*, \text{ if } x \times y \in mJ.$$

A collineation of PJ is a pair of bijections of the set of points to itself and the set of lines to itself which preserves the defining relations. $W \in \Gamma$ induces a collineation PW by PJ by $PW(x_*) = (Wx)_*$ and $PW(x^*) = (W^{*-1}x)^*$ [Lemma 1.7]. For a subgroup H of Γ, let $PH = \{PW \mid W \in H\}$.

$T(V_{x,y}z, w) = T(z, V_{y,x}w)$ for $w, x, y, z \in J$, by the proof of Lemma 1.8(4). Since $T(z, w)$ is nondegenerate, $V_{x,y} = 0$ if and only if $V_{y,x} = 0$. Thus the map from PJ to itself interchanging x_* and x^* preserves the defining relations. This establishes the principle of duality, that any theorem about PJ remains true when points and lines are interchanged.

Let T' be the group generated by $T_{a[ij],e_j}$, $a \in \mathfrak{O}$, $1 \leqslant i < j \leqslant 3$. If I is an ideal of R, let T_I be the group generated by $T_{x,y}$, $x \in IJ$, $y \in \Pi$, $T(x, y) = 0$. $T' \subseteq S$, $T_I \subseteq S$, and T_I is a normal subgroup of G [Lemma 1.8].

Let I be an ideal of R. Write $x_* \equiv y_* \pmod I$ and $x^* \equiv y^* \pmod I$ if $Rx + IJ = Ry + IJ$.

PROPOSITION 2.1. (1) *If* $x_* \equiv y_* \pmod I$, *there is* $\phi \in T' \cap T_I$ *such that* $P\phi x_* = y_*$.

(2) *If* $x_{1*} \nsim x_{2*}$, $y_{1*} \nsim y_{2*}$, *and* $x_{i*} \equiv y_{i*} \pmod I$, *there is* $\phi \in T_I$ *such that* $P\phi x_{i*} = y_{i*}$.

(3) *If* $T(x_1 \times x_2, x_3) \in R - m$, $T(y_1 \times y_2, y_3) \in R - m$, *and* $x_{i*} \equiv y_{i*} \pmod I$, *there is* $\phi \in T_I$ *such that* $P\phi x_{i*} = y_{i*}$.

(4) (1)–(3) *hold with points replaced by lines.*

PROOF. (1) We first prove that for any $z \in \Pi$ there is $\phi \in T'$ such that $P\phi e_{1*} = z_*$. Write $z = \Sigma\, \alpha_i e_i + \Sigma\, a_i[jk]$. If all $\alpha_i \in m$, then some $a_i \in \mathfrak{O} - m\mathfrak{O}$ and there is $b \in \mathfrak{O}$ with $n(b, a_i) \in R - m$. Replacing z by $T_{b[jk],e_j} z$, we can assume that some α_i is a unit. Replacing z by $\alpha_i^{-1} z$, we can assume that $\alpha_i = 1$. Then

$$T_{a_j[ki],e_i} T_{a_k[ij],e_i} e_i = z, \tag{iv}$$

since $z^\# = 0$. The claim follows, since

$$T_{1[i1],e_1} e_1 = T_{\gamma_1^{-1}\gamma_i^{-1}[i1],e_i} \gamma_1 \gamma_i e_i, \qquad i \neq 1.$$

Next we show that, if $z_* \equiv e_{1*} \pmod I$, $I \neq R$, there is $\psi \in T' \cap T_I$ such that $P\psi e_{1*} = z_*$. Write $z = \Sigma\, \alpha_i e_i + \Sigma\, a_i[jk]$, $a_i \in I\mathfrak{O}$. Since $I \neq R$, α_1 is a unit and we can assume that $\alpha_1 = 1$. Taking $i = 1$ in (iv) establishes the lemma.

Now we prove (1). If $I = R$, (1) holds by the first paragraph. Assume $I \neq R$. By the first paragraph, there is $\phi \in T'$ such that $P\phi x_* = e_{1*}$. Then $P\phi y_* \equiv e_{1*} \pmod I$, so, by the second paragraph, there is $\psi \in T' \cap T_I$ such that $P\psi e_{1*} = P\phi y_*$. Then $P(\phi^{-1}\psi\phi)x_* = y_*$ and $\phi^{-1}\psi\phi \in T' \cap T_I$ [Lemma 1.8(3)], as required.

(2) First assume that $x_{1*} = e_{1*} = y_{1*}$ and $x_{2*} = e_{2*}$. Let $y_2 = \Sigma\, \alpha_i e_i + \Sigma\, a_i[jk]$, $a_i \in I\mathfrak{O}$. If α_2 is a unit, we can assume that $\alpha_2 = 1$. (2) follows in this case by taking $i = 2$ and $z = y_2$ in (iv), since T_{p,e_2} fixes e_1 for any p. If $\alpha_2 \in m$, then $I = R$. Since $e_{1*} \nsim y_{2*}$, either $\alpha_3 \in R - m$ or $a_1 \in \mathfrak{O} - m\mathfrak{O}$. We reduce to the case $\alpha_2 \in R - m$ by applying $T_{b[23],e_3}$ to y_2 for $b \in \mathfrak{O}$ such that $n(b)\alpha_3 + n(b, a_1)$ is a unit.

We now prove (2) in general. By (1), we can assume that $x_{1*} = y_{1*}$. Applying (1) again shows that there is $\phi \in T'$ such that $P\phi x_{1*} = e_{1*} = P\phi y_{1*}$. By the preceding paragraph, there is $\psi \in T_R$ such that $P\psi$ fixes e_{1*} and $P\psi(P\phi x_{2*}) = e_{2*}$. Applying the last paragraph again, there is $\tau \in T_I$ such that $P\tau$ fixes e_{1*} and $P\tau e_{2*} = P\psi\phi y_{2*}$. Then $\eta = \phi^{-1}\psi^{-1}\tau\psi\phi \in T_I$ and $P\eta x_{i*} = y_{i*}$.

(3) First assume that $x_{1*} = e_{1*} = y_{1*}$, $x_{2*} = e_{2*} = y_{2*}$, and $x_{3*} = e_{3*}$. Let $y_3 = \Sigma\alpha_i e_i + \Sigma a_i[jk]$, $\alpha_3 \in R - m$, $a_i \in I\mathfrak{O}$. We can assume that $\alpha_3 = 1$, so it suffices to take $i = 3$ and $z = y_3$ in (iv).

We now prove (3) in general. By (2), we can assume that $x_{1*} = y_{1*}$ and

$x_{2*} = y_{2*}$. Applying (2) again, there is $\phi \in T_R$ such that $P\phi x_{i*} = e_{i*} = P\phi y_{i*}$, $i = 1, 2$. By the preceding paragraph, there is $\psi \in T_R$ such that $P\psi(P\phi x_{3*}) = e_{3*}$ and $P\psi$ fixes e_{1*} and e_{2*}. Again by the preceding paragraph, there is $\tau \in T_I$ such that $P\tau e_{3*} = P\psi\phi y_{3*}$ and $P\tau$ fixes e_{1*} and e_{2*}. Then $\eta = \phi^{-1}\psi^{-1}\tau\psi\phi \in T_I$ and $P\eta x_{i*} = y_{i*}$.

(4) Let L' be the group generated by $T_{e_j,p[ij]}$ for $p \in \mathfrak{O}$, and let L_I be the group generated by $T_{y,x}$, $x \in IJ, y \in \Pi$, $T(x, y) = 0$. By the duals of (1)–(3), it suffices to show that $L' = T'$ and $L_I = T_I$. $L' = T'$ because $T_{e_j,p[ij]} = T_{p[ij],e_i}$, by (ii). Consider $T_{y,x}$, $x \in IJ, y \in \Pi$, $T(x, y) = 0$. By (1) there is $\phi \in T'$ such that $\phi T_{y,x}\phi^{-1} = T_{e_1,z}$, $z \in IJ$. We can assume that $z = p[12] + q[31], p, q \in I\mathfrak{O}$, as in (iii). By [5, p. 49],

$$T_{e_1,z} = T_{e_1,p[12]}T_{e_1,q[31]} = T_{p[12],e_2}T_{q[31],e_3} \in T_I,$$

so $T_{y,x} = \phi^{-1}T_{e_1,z}\phi \in T_I$. Then $L_I \subseteq T_I$. □

Direct verification shows that $x_*|e_i^*$ if and only if $x \in J_0(e_i)$. By duality, $e_{i*}|x^*$ if and only if $x \in J_0(e_i)$.

LEMMA 2.2. *If $a_* \nsim b_*$, $(a \times b)^*$ is the unique line on both a_* and b_*. Dually, if $x^* \nsim y^*$, $(x \times y)_*$ is the unique point on both x^* and y^*.*

PROOF. By Proposition 2.1 and Lemma 1.7, we can assume that $a_* = e_{1*}$ and $b_* = e_{2*}$. Since $e_{i*}|x^*$ if and only if $x \in J_0(e_i)$, the lemma follows. □

Even when R is a field, if $\mathfrak{O}$ is not a division algebra, there are pairs of points which are on more than one line. However, when R is a field, any two points are on at least one line [5, pp. 35, 50]. We note that this no longer holds when R is a local ring. For example, let $R = F[x, y]_{(x,y)}$, the localization of the polynomial ring $F[x, y]$ at the maximal ideal (x, y). $R \subset Q = F(x, y)$, the field of rational functions. Let $\mathfrak{O}$ be an octonion algebra over R and let $J = H(\mathfrak{O}_3, 1)$. Let $a = e_1$ and $b = e_1 + x^2e_2 + y^2e_3 + x[12] + y[31] + xy[23]$. Assume c^* is on both a_* and b_* in PJ. $a \times b = y^2e_2 + x^2e_3 - xy[23]$. Applying Lemma 2.2 to the images of a, b, and c in $J \otimes Q$ yields $Qc = Q(a \times b)$. Thus there is $\alpha \in Q$ such that $y^2e_2 + x^2e_3 + xy[23] = \alpha c$, $c \in J - mJ$, which is impossible.

LEMMA 2.3. *If a_*, $x^* \in PJ$, there is y^* such that $a_*|y^*$ and $x^* \nsim y^*$.*

PROOF. We can assume that $a_* = e_{1*}$ [Proposition 2.1(1)]. We can take $y^* = e_2^*$ or e_3^* unless x^* is connected to both, whence $x \equiv a_1[23] \pmod{mJ}$, $a_1 \in \mathfrak{O} - m\mathfrak{O}$. In this case, there is $p \in \mathfrak{O}$ with $n(a_1, p) \in R - m$, and we take $y = e_2 + p[23] + \gamma_2\gamma_3 n(p)e_3$. □

LEMMA 2.4. *Let $a_*|x^*$, $a_* \nmid y^*$, and $x^* \nsim y^*$. Then there is $q \in J$, $T(q, x) = 0$, such that $PT_{q,x}$ fixes all points on x^* and all lines on a_* and does not fix y^*.*

PROOF. There is z^* such that $a_*|z^*$ and $z^* \nsim x^*$ [Lemma 2.3]. By Proposition 2.1(4), we can assume that $x^* = e_3^*$ and $z^* = e_2^*$, so $a_* = e_{1*}$. For $p \in \mathfrak{O}$, $PT_{p[31],e_3}$ fixes every point b_* on e_3^*, since $b \in J_0(e_3)$. Likewise $PT_{p[31],e_3}$ fixes every line on e_{1*}, since

$$T_{p[31],e_3}^{*-1} = T_{e_3,-p[31]} = T_{-p[31],e_1}$$

by (ii) and Lemma 1.8. Thus it suffices to find $p \in \mathfrak{O}$ such that $PT_{p[31],e_3}y^* \neq y^*$. If $y = \sum \alpha_i e_i + a_i[jk]$,

$$T^{*-1}_{p[31],e_3}\, y = y + \gamma_1\gamma_3(\alpha_1 n(p) - n(a_2, p))e_3 - \alpha_1 p[31] - \gamma_1(pa_3)^d[23].$$

$\alpha_1 \in R - m$, $\alpha_2 \in R - m$, or $a_3 \in \mathfrak{O} - m\mathfrak{O}$, since $e_3^* \nsim y^*$. Since $T^{*-1}_{p[31],e_3}$ does not change the e_1, e_2, or [12] components of y, it suffices to find $p \in \mathfrak{O}$ such that $T^{*-1}_{p[31],e_3}\, y \neq y$. Since $e_{1*} \nmid y_*$, either α_1, a_2, or a_3 is nonzero. If α_1 or a_3 is nonzero, take $p = 1$; otherwise, take p such that $n(a_2, p) \neq 0$. □

LEMMA 2.5. *Let $a_*|x^*$, $b_*|x^*$, $a_* \neq b_*$. Then there is $q \in J$, $T(q, x) = 0$, such that $PT_{q,x}$ fixes all points on x^* and all lines on a_* and does not fix all lines on b_*.*

PROOF. There is $y^* \in PJ$ such that $b_*|y^*$ and $x^* \nsim y^*$ [Lemma 2.3]. Then $a_* \nmid y^*$, else $a_* = (x \times y)_* = b_*$, so we are done by Lemma 2.4. □

3. Existence of transvections in subgroups normalized by S. We prove that a subgroup of Γ which is normalized by S and not contained in $R - m$ contains T_I for a nonzero ideal I of R. We argue by repeatedly taking commutators of elements of N with elements of T_R and applying the geometry of §2. Let $[\phi, \psi] = \phi\psi\phi^{-1}\psi^{-1}$ for $\phi, \psi \in \Gamma$.

THEOREM 3.1. *Let N be a subgroup of Γ such that N is normalized by S and $PN \neq 1$. Then N contains $T_{p[12],e_1}$ for some $p \neq 0$.*

PROOF. We first show that there are $\phi \in N$, b_*, and y^* such that $P\phi y^* \neq y^*$ and both y^* and $P\phi y^*$ are on b_*. Proposition 2.1 and Lemma 2.2 imply that every line of PJ has the form $(c \times d)^*$ for some c_* and d_*, so only the identity collineation fixes every point of PJ. Take $\eta \in N$ and a_* such that $P\eta a_* \neq a_*$. Since $P\eta a_*$ has the form $(y_1 \times y_2)_*$ for some y_i^*, there is y^* such that $P\eta a_*|y^*$ and $a_* \nmid y^*$. Take x^* such that $a_*|x^*$ and $x^* \nsim y^*$ [Lemma 2.3]. By Lemma 2.4, there is $\tau \in S$ such that $P\tau$ fixes all points on x^* and all lines on a_* and does not fix y^*. Let $\phi = [\tau, \eta] \in N$. Since $P\eta\tau^{-1}\eta^{-1}$ fixes all lines on $P\eta a_*$, $P\phi y^* = P\tau y^* \neq y^*$. Since $x^* \nsim y^*$, $b_* = (x \times y)_*$ is on both x^* and y^*. Since $P\tau$ fixes all points on x^*, $b_* = P\tau b_*$ is on $P\tau y^* = P\phi y^*$, as required.

Take ϕ, y^*, and b_* as above. By the dual of Lemma 2.5, there is $\zeta \in S$ such that $P\zeta$ fixes all lines on b_* and all points on y^* and does not fix all points on $P\phi y^*$. Let $\psi = [\zeta, \phi] \in N$. $P\psi$ fixes $P\phi y^*$, since $P\zeta$ fixes all lines on b_*. $P\psi$ does not fix all points on $P\phi y^*$, since $P\phi\zeta^{-1}\phi^{-1}$ fixes all such points and $P\zeta$ does not. Take $\theta \in S$ such that $P\theta(P\phi y^*) = e_1^*$ and set $\xi = \theta\psi\theta^{-1}$. Then $\xi \in N$, $P\xi$ fixes e_1^*, and $P\xi$ does not fix some c_* on e_1^*.

By Lemma 2.5, there is $q \in J$, $T(q, e_1) = 0$ such that PT_{q,e_1} fixes every point on e_1^* and every line on c^* and does not fix every line on $P\xi c_*$. Let $\chi = [T_{q,e_1}, \xi] \in N$. $P\chi \neq 1$, since $P\xi T_{q,e_1}\xi^{-1}$ fixes every line on $P\xi c_*$ and PT_{q,e_1} does not.

$$\chi = T_{q,e_1}T_{-\xi q,\xi^{*-1}e_1} = T_{q,e_1}T_{r,e_1} = T_{q+r,e_1}$$

for $r \in J$, $T(r, e_1) = 0$ [5, p. 49]. By (iii), $\chi = T_{s[12]+t[31],e_1}$ for $s, t \in \mathfrak{O}$ not both zero. If $t = 0$, we are done. If $t \neq 0$, N contains

$$\left[T_{1[23],e_3}, T_{s[12]+t[31],e_1}\right] = T_{\gamma_3 t^d[12],e_1}. \quad \square$$

Lemmas 3.2 and 3.3 below show that the hypothesis $PN \neq 1$ of Theorem 3.1 is equivalent to $N \not\subseteq R - m$. Lemmas 3.4 and 3.5 show that the conclusion of Theorem 3.1 implies that N contains T_I for a nonzero ideal I of R. These lemmas are also needed for later use.

LEMMA 3.2. (1) *If $\phi \in \Gamma$ fixes all Re_i, then ϕ fixes all $\mathfrak{O}[jk]$.*

(2) *If $\phi \in \Gamma$ fixes Re_1 and interchanges Re_2 and Re_3, then ϕ fixes $\mathfrak{O}[23]$ and interchanges $\mathfrak{O}[12]$ and $\mathfrak{O}[31]$.*

PROOF. (1) We note that $Re_i + J_{1/2}(e_i)$ is the radical of the symmetric form $(x, y) \to T(e_i \times x, y)$ on J. Then ϕ fixes

$$\left[Re_j + J_{1/2}(e_j)\right] \cap \left[Re_k + J_{1/2}(e_k)\right] = \mathfrak{O}[jk].$$

(2) is proved similarly. □

LEMMA 3.3. $P\Gamma \cong \Gamma/(R - m)$.

PROOF. We show that $R - m$ is the kernel of the homomorphism $\phi \to P\phi$ of Γ onto $P\Gamma$. Clearly $P(R - m) = 0$. Conversely, suppose that $P\phi = 1$, $\phi \in \Gamma$. Since ϕ fixes each Re_i, ϕ fixes each $\mathfrak{O}[jk]$ [Lemma 3.2(1)]. Let $\phi(\gamma_i e_i) = \beta_i \gamma_i e_i$, $\beta_i \in R - m$. Since ϕ fixes $R(\Sigma\, \gamma_i e_i + \Sigma\, 1[jk])$, $\beta_1 = \beta_2 = \beta_3$. For $a \in \mathfrak{O}$, since ϕ fixes $R(\gamma_i e_i + a[ij] + \gamma_j n(a) e_j)$, we have $\phi(a[ij]) = \beta_1 a[ij]$. Then ϕ is linear, so it is multiplication by β_1. □

LEMMA 3.4. *Let N be a subgroup of Γ normalized by S. Set*

$$L = \{p \in \mathfrak{O} \mid T_{p[12],e_1} \in N\}.$$

Then $L = I\mathfrak{O}$ for an ideal I of R.

PROOF. L is an additive group [5, p. 49]. If $p \in L$ and $q \in \mathfrak{O}$, N contains

$$\left[T_{q[23],e_2}, T_{p[12],e_1}\right] = T_{\gamma_2 pq[13],e_1}. \tag{v}$$

Then N contains

$$\left[T_{1[23],e_3}, T_{\gamma_2 pq[13],e_1}\right] = T_{\gamma_2\gamma_3 pq[12],e_1}, \tag{vi}$$

so L is a right ideal. We are done by Lemma 1.12. □

LEMMA 3.5. *T_I is generated by $\phi T_{a[12],e_1}\phi^{-1}$, $\phi \in T'$, $a \in I\mathfrak{O}$.*

PROOF. Consider $T_{x,y}$, $x \in IJ, y \in \Pi$, $T(x, y) = 0$. There is $\psi \in T'$ such that

$$\psi T_{x,y}\psi^{-1} = T_{p[12]+q[13],e_1} = T_{p[12],e_1} T_{q[13],e_1}$$

for $p, q \in I\mathfrak{O}$. We are done by (v). □

COROLLARY 3.6. *Let N be a subgroup of Γ normalized by S and not contained in $R - m$. Then N contains T_I for a nonzero ideal I of R.* □

4. Generators of congruence subgroups of orthogonal groups. If Q is a quadratic form on an R-module M, the orthogonal group $\mathfrak{O}(M)$ is the group of module automorphisms ϕ of M such that $Q(\phi x) = Q(x)$ for $x \in M$. If $a \in M$ and $Q(a)$ is a unit, define the hyperplane reflection $S_a \in \mathfrak{O}(M)$ by $S_a(x) = x - Q(a)^{-1}Q(x, a)a$. For an ideal I of R, let $\mathfrak{O}(M, I)$ be the congruence subgroup of

level I, the kernel of the homomorphism $\mathfrak{O}(M) \to \mathfrak{O}(M/IM)$. A hyperbolic plane is a free R-module $Ru \oplus Rv$ with the quadratic form $Q(\alpha u + \beta v) = \alpha\beta$.

Let Q be a nondegenerate quadratic form on an (R, m)-module M containing a hyperbolic plane. We prove that $\mathfrak{O}(M)$ is generated by hyperplane reflections except in one case and that $\mathfrak{O}(M, I)$ is generated by pairs of hyperplane reflections for $I \neq R$.

LEMMA 4.1. *Let Q be a nondegenerate quadratic form on a finite-dimensional vector space V over a field F. Assume that either F has more than two elements or V is not a hyperbolic plane. Then V has a basis $x_1, \ldots, x_n$ such that each $Q(x_i) \neq 0$.*

PROOF. We can clearly assume that the characteristic of F is 2. It suffices to show that V has a nonzero subspace X such that Q is nondegenerate on X and X has a basis $\{x_i\}$ with $Q(x_i) \neq 0$; if so, $V = X \oplus X^{\perp}$ and, for any $y \in X^{\perp}$, either $Q(y) \neq 0$ or $Q(x_1 + y) \neq 0$. Take $u \in V$ such that $Q(u) \neq 0$. If there is $w \in V$ such that $Q(u, w) \neq 0$ and $Q(w) \neq 0$, we can take $X = Fu + Fw$. Thus we can assume that $Q(w) = 0$ for all $w \in V$ such that $Q(u, w) \neq 0$. Choose $v \in V$ such that $Q(u, v) = 1$. $Q(v) = 0$ and Q is nondegenerate on $Y = Fu + Fv$. If $z \in Y^{\perp}$, then $Q(u, v + z) = 1$, so $0 = Q(v + z) = Q(z)$. Since Q is nondegenerate, $Y^{\perp} = 0$ and $V = Fu + Fv$. Then F has more than two elements, so we can take $\alpha \in F - 0$ such that $\alpha \neq -Q(u)$. Then $Q(u, u + \alpha v) = \alpha \neq 0$, and $Q(u + \alpha v) = Q(u) + \alpha \neq 0$, a contradiction. □

The first paragraph of the following theorem is proved by Klingenberg for char $R/m \neq 2$ in [**9**]. When the characteristic of R/m is arbitrary, the first paragraph is related to [**8**, Theorem I].

THEOREM 4.2. *Let Q be a nondegenerate quadratic form on a free module M of finite rank over (R, m) that contains a hyperbolic plane. Assume that either R/m has order greater than two or M/mM is not the direct sum of two hyperbolic planes. Then $\mathfrak{O}(M)$ is generated by hyperplane reflections.*

Moreover, let I be an ideal of R, $I \neq R$. Let $Ru \oplus Rv \subseteq M$ be a hyperbolic plane. Then $\mathfrak{O}(M, I)$ is generated by $(S_b S_c) S_a S_{u+v} (S_b S_c)^{-1}$, $a \equiv u + v \pmod{IM}$. In particular, $\mathfrak{O}(M, I)$ is generated by $S_p S_q$, $p \equiv q \pmod{IM}$.

PROOF. Let $\phi \in \mathfrak{O}(M)$. By the hypotheses on M/mM and R/m, the image of ϕ in $\mathfrak{O}(M/mM)$ equals $S_{b_1} \cdots S_{b_t}$, $b_i \in M/mM$, $0 \neq Q(b_i) \in R/m$ [**3**, p. 19]. If $a_i \in M$ is a preimage of b_i, $Q(a_i)$ is a unit and $(S_{a_1} \cdots S_{a_t})^{-1} \phi \in \mathfrak{O}(M, m)$. Thus it suffices to establish the second paragraph.

Let $\phi \in \mathfrak{O}(M, I)$. Set $N = (Ru + Rv)^{\perp}$, so $M = Ru \oplus Rv \oplus N$ and Q restricted to N is nondegenerate. N has a free basis $y_1, \ldots, y_n$ such that $Q(y_i)$ is a unit, by Lemma 4.1 and the remark before Lemma 1.9. Let $x_1, \ldots, x_n$ be a dual basis of N, so $Q(x_i, y_j) = \delta_{ij}$ the Kronecker delta.

We prove by induction on s that, for $1 \leqslant s \leqslant n$, there are $G_1, \ldots, G_s$ such that $\phi_s = G_s \cdots G_1 \phi$ fixes $x_1, \ldots, x_s$ and G_i is a product of terms of the form $(S_b S_c) S_{u+v} S_a (S_b S_c)^{-1}$, $a \equiv u + v \pmod{IM}$. Assume we have found ϕ_{s-1}, taking $\phi_0 = \phi$. $\phi_{s-1} \in \mathfrak{O}(M, I)$ and ϕ_{s-1} fixes $x_1, \ldots, x_{s-1}$.

If $\lambda \in R$, $\lambda \equiv 1 \pmod I$, set $\zeta_\lambda = S_{u+v}S_{\lambda u+v} \in \mathfrak{D}(M, I)$. $\zeta_\lambda(u) = \lambda^{-1}u$, $\zeta_\lambda(v) = \lambda v$, and ζ_λ fixes the elements of N.

Set $\tau = S_{Q(y_s)u+y_s}S_{y_s} \in \mathfrak{D}(M)$ (since $Q(y_s)$ is a unit). $\tau(x_s) = u + x_s$ and τ fixes x_i for $i \neq s$.

Since $\phi_{s-1}(x_s) \equiv x_s \pmod{IM}$, $\tau\phi_{s-1}(x_s) \equiv u + x_s \pmod{IM}$, so $\tau\phi_{s-1}(x_s) = \alpha u + \xi v + x$, $\alpha \equiv 1 \pmod I$, $\xi \in I$, $x \in N$, $x \equiv x_s \pmod{IN}$. Then

$$\zeta_\alpha\tau\phi_{s-1}(x_s) = u + \beta v + x, \qquad \beta \in I. \tag{vii}$$

Let $z = x - x_s \in IN$. $Q(x_s) = Q(u + \beta v + x) = \beta + Q(x)$, whence

$$Q(u + v + z, u + \beta v + x) = Q(u + v + z)$$

follows by substituting $z = x - x_s$. This implies that

$$\psi(u + \beta v + x) = (1 - \beta)u + x_s \quad \text{for } \psi = S_{u+v}S_{u+v+z}.$$

ψ fixes x_i for $i < s$, since

$$Q(z, x_i) = Q(x, x_i) = Q(u + \beta v + x, x_i) = Q(x_s, x_i) = 0$$

by (vii). Hence $G_s = \tau^{-1}\zeta_{1-\beta}\psi\zeta_\alpha\tau$ has the required form and $\phi_s = G_s\phi_{s-1}$ fixes $x_1, \ldots, x_s$, completing the induction. $\phi_n = G_n \cdots G_1\phi$ fixes the elements of N and fixes $N^\perp = Ru + Rv$. Let $\phi_n(v) = \alpha u + \beta v$, $\alpha \in I$, $\beta \equiv 1 \pmod I$. Since $0 = Q(v) = Q(\phi_n(v)) = \alpha\beta$, $\alpha = 0$ and $\phi_n(v) = \beta v$. Similarly, $\phi_n(u) = \beta^{-1}u$, so $\phi_n = \zeta_\beta$. Thus $\phi = G_1^{-1} \cdots G_n^{-1}\zeta_\beta$. □

5. Generating the subgroup of S_I fixing e_1 and $J_0(e_1)$. If I is an ideal of R and H is a subgroup of Γ, let

$$H_I = \{\phi \in H | \phi(x) \equiv x \pmod{IJ}, x \in J\}.$$

In this section we apply Theorem 4.2 to determine the elements of S_I fixing e_1 and $J_0(e_1)$. This was done in [5, Theorem 2.10] when R is a field. More generally, we determine the elements of G_I that fix Re_1 and $J_0(e_1)$ and preserve the quadratic form $T(x^\sharp)$ on $J_0(e_1)$.

Let Q be a quadratic form on a finite free R-module M. Let $C(Q)$ be the corresponding Clifford algebra, i.e., the tensor algebra on M modulo the ideal generated by $v \otimes v - Q(v)1$, $v \in M$. If M has a free basis $w_1, \ldots, w_n$, $C(Q)$ has a free basis 1 and $w_{i_1} \cdots w_{i_t}$, $i_1 < \cdots < i_t$, $1 \leqslant t \leqslant n$, and we can identify $M \subset C(Q)$. $C(Q)$ has the canonical involution π fixing the elements of M.

Let $M' = \{v \in M | Q(v) \in R - m\}$. Let $\Gamma^e(Q)$ be the multiplicative group in $C(Q)$ generated by v_1v_2, $v_i \in M'$. The map of $C(Q)$ to itself taking x to xx^π induces a homomorphism λ: $\Gamma^e(Q) \to R - m$ such that $\lambda(v_1v_2) = Q(v_1)Q(v_2)$, $v_i \in M'$. Let $\mathrm{Spin}(Q)$ be the kernel of λ.

If $v \in M'$, $v^{-1} = Q(v)^{-1}v$ in $C(Q)$ and $vxv^{-1} = -S_v x$ for $x \in M$. Define a homomorphism χ: $\Gamma^e(Q) \to \mathfrak{D}(M)$ by $[\chi(u)]x = uxu^{-1}$, $u \in \Gamma^e(Q)$ and $x \in M$; $\chi(v_1 \cdots v_{2r}) = S_{v_1} \cdots S_{v_{2r}}$ for $v_i \in M'$.

Write e_1 as e, $J_i(e_1)$ as J_i, and $1 - e_1$ as f. Take $Q(x) = T(x^\sharp)$ on J_0. One sees directly that Q is nondegenerate, $N(e + x) = Q(x)$, and $x^\sharp = Q(x)e$, for $x \in J_0$. Set $x' = e \times x$ for $x \in J_0$; $x' = -S_f x$ (with respect to Q) and $x'' = x$.

Linearizing $x^{\#\#} = N(x)x$ yields

$$(x \times z) \times y^{\#} + (x \times y) \times (z \times y) = T(y^{\#}, x)z + T(y^{\#}, z)x + T(x \times z, y)y.$$

Then, if $v \in J_0$ and $z \in J_{1/2}$, $(e \times v) \times (z \times v) = T(v^{\#}, e)z = Q(v)z$, since $e \times J_{1/2} = 0$ by examination and $v^{\#} = Q(v)e$ is orthogonal to $J_{1/2}$. $V_v z = U_{v,1}z = U_{v,e}z$, by QJ9 and the Peirce identities [5, pp. 6–7]. Thus $V_v V_{v'}z = U_{v,e}U_{v',e}z = (e \times v) \times (v \times z)$ (by the definition of U and the orthogonality of the Peirce spaces) $= Q(v)z$. Let $C^e(Q)$ be the subalgebra of $C(Q)$ generated by v_1v_2, $v_i \in J_0$. It follows that there is a homomorphism $\rho\colon C^e(Q) \to \mathrm{End}_R(J_{1/2})$ such that $\rho(v_1v_2) = V_{v_1}V_{v_2'}$ (since the tensor algebra is graded into even and odd components and $v \otimes v - Q(v)1$ is contained in the even component).

If $x, v \in J_0$, $T(v, x) = T(v, e \times x') = T(e, x' \times v) = T(e, Q(x', v)e) = Q(x', v)$. If $v \in J_0'$, $U_v x = T(v, x)v - v^{\#} \times x = Q(v, x')v - Q(v)x' = Q(v)S_vS_fx$.

Combining the last two paragraphs with the Peirce identities shows that, for $v \in J_0$, $U_{e+v}(e) = e$, $U_{e+v}|J_{1/2} = V_v$, and $U_{e+v}|J_0 = Q(v)S_vS_f$. If $u = v_1 \cdots v_{2r} \in \Gamma^e(Q)$ for $v_i \in J_0'$, let

$$W_u = U_{e+v_1}U_{e+v_2'} \cdots U_{e+v_{2r-1}}U_{e+v_{2r}'}.$$

$W_u(e) = e$, $W_u|J_{1/2} = u^{\rho}$, and $W_u|J_0 = \lambda(u)u^{\chi}$. Thus W_u is independent of the form $u = v_1 \cdots v_{2r}$ and $u \to W_u$ is a well-defined homomorphism from $\Gamma^e(Q)$ to G [Corollary 1.5].

LEMMA 5.1. *Let $\phi \in G$ induce the identity map on J_0 and satisfy $\phi(e) = \alpha e$, $\alpha \in R - m$. Then $\phi|J_{1/2}$ is multiplication by τ, for $\tau \in R - m$ such that $\tau^2 = \alpha$.*

PROOF. Taking $a = 1$ shows that $N(\phi a) = \alpha N(a)$, $a \in J$. By Lemma 3.2(1), $\phi(a_2[31]) = b_2[31]$ and $\phi(a_3[12]) = b_3[12]$ where $a_2 \to b_2$ and $a_3 \to b_3$ are bijections of $\mathfrak{D}$. For $a_i \in \mathfrak{D}$,

$$\begin{aligned}\alpha\gamma_1\gamma_2\gamma_3 t(a_1a_2a_3) &= \alpha N(a_1[23] + a_2[31] + a_3[12]) \\ &= N(\phi(a_1[23] + a_2[31] + a_3[12])) \\ &= N(a_1[23] + b_2[31] + b_3[12]) \\ &= \gamma_1\gamma_2\gamma_3 t(a_1b_2b_3).\end{aligned}$$

Then $b_2b_3 = \alpha a_2a_3$, since $t(xy) = n(x, y^d)$ is nondegenerate. Taking a_3 so $b_3 = 1$ gives $b_2 = a_2c$ for $c \in \mathfrak{D}$, so $(a_2c)b_3 = \alpha a_2a_3$. c is invertible, since the surjectivity of $a_2 \to a_2c$ gives $n(c) \in R - m$ and $c^{-1} = n(c)^{-1}c^d$. Setting $a_2 = c^{-1}$ gives $b_3 = \alpha c^{-1}a_3$, so $(a_2c)(c^{-1}a_3) = a_2a_3$. Replacing a_3 by ca_3 yields $(a_2c)a_3 = a_2(ca_3)$. Then $c \in R - m$ [Lemma 1.11] and $\phi(a_2[31]) = ca_2[31]$.

$$\begin{aligned}-\gamma_1\gamma_3\alpha &= \alpha N(e_2 + 1[31]) = N(\phi(e_2 + 1[31])) \\ &= N(e_2 + c[31]) = -\gamma_1\gamma_3c^2,\end{aligned}$$

so $c^2 = \alpha$ and $\phi(a_3[12]) = \alpha c^{-1}a_3[12] = ca_3[12]$. □

LEMMA 5.2. *There is no $\phi \in G$ such that ϕ fixes Re, ϕ is the identity map on $\mathfrak{D}[23]$, and ϕ interchanges e_2 and e_3.*

PROOF. Assume such ϕ exists. Let $\phi e = \alpha e$, $\alpha \in R$. Applying ϕ to $e + 1[23]$ shows that $N(\phi x) = \alpha N(x)$ for $x \in J$. By Lemma 3.2(2), $\phi(a_2[31]) = b_2[12]$ and $\phi(a_3[12]) = b_3[31]$, where $a_2 \to b_2$ and $a_3 \to b_3$ are bijections of $\mathfrak{O}$. It follows as in the proof of Lemma 5.1 that $b_3 b_2 = \alpha a_2 a_3$. Taking a_3 so $b_3 = 1$ shows that $b_2 = \alpha a_2 c$ for invertible $c \in \mathfrak{O}$, so $b_3(a_2 c) = a_2 a_3$. Taking $a_2 = c^{-1}$ yields $b_3 = c^{-1}a_3$, so $(c^{-1}a_3)(a_2 c) = a_2 a_3$. Taking $a_2 = 1$ gives $c^{-1}a_3 c = a_3$, so $c \in R - m$ [Lemma 1.11] and $a_3 a_2 = a_2 a_3$, a contradiction. □

Henceforth let I be an ideal of R. Let

$$G_e = \{\phi \in G | \phi \text{ fixes } Re \text{ and } J_0 \text{ and } \phi|J_0 \in \mathfrak{O}(J_0)\},$$

and let $G_{e,I} = G_e \cap G_I$ and $S_{e,I} = G_e \cap S_I$. Let $\Gamma^e(Q, I)$ be the subgroup of $\Gamma^e(Q)$ generated by $(a_1 a_2)a_3 f(a_1 a_2)^{-1}$, $a_i \in J_0'$, $a_3 \equiv f \pmod{IJ_0}$. Let $\mathrm{Spin}(Q, I) = \mathrm{Spin}(Q) \cap \Gamma^e(Q, I)$.

THEOREM 5.3. *$u \to \lambda(u)^{-1}W_u$ is an isomorphism of $\Gamma^e(Q, I)$ onto $G_{e,I}$.*

PROOF. Since $\lambda(u)^{-1}W_u|J_0 = u^\chi \in \mathfrak{O}(J_0)$, $u \to \lambda(u)^{-1}W_u$ is a homomorphism from $\Gamma^e(Q, I)$ to $G_{e,I}$. We claim that $G_{e,I}|J_0 = \Gamma^e(Q, I)^\chi$. Since $\lambda(u)^{-1}W_u|J_0 = u^\chi$,

$$\Gamma^e(Q, I)^\chi \subseteq G_{e,I}|J_0 \subseteq \mathfrak{O}(J_0, I).$$

If $I \neq R$, $\Gamma^e(Q, I)^\chi = \mathfrak{O}(J_0, I)$ [Theorem 4.2], proving the claim. If $I = R$, Theorem 4.2 implies that $\Gamma^e(Q)^\chi$ has index at most two in $\mathfrak{O}(J_0)$. Since $G_e|J_0 \neq \mathfrak{O}(J_0)$ [Lemma 5.2], $\Gamma^e(Q)^\chi = G_e|J_0$, as claimed.

Let $\phi \in G_{e,I}$. By the last paragraph, there is $u \in \Gamma^e(Q, I)$ such that $\phi^{-1}\lambda(u)^{-1}W_u$ is the identity map on J_0. $\phi^{-1}\lambda(u)^{-1}W_u(e) = \alpha e$, $\alpha \in R - m$. By Lemma 5.1, $\phi^{-1}\lambda(u)^{-1}W_u|J_{1/2}$ is multiplication by $\tau \in R - m$ such that $\tau^2 = \alpha$. $\tau \equiv 1 \pmod{I}$ and $\phi = \lambda(\tau u)^{-1}W_{\tau u}$, $\tau u \in \Gamma^e(Q, I)$, so the homomorphism is onto.

Let u be in the kernel of the homomorphism. Since $\lambda(u)^{-1}W_u|J_0 = u^\chi$ is the identity, u is in the center of $C(Q)$. $C(Q)/mC(Q)$ is central simple over R/m, since it is isomorphic to the Clifford algebra for $T(x^\sharp)$ on J_0/mJ_0 [3, p. 42]. Then R is the center of $C(Q)$, by the associative analogue of [2, Theorem 1.8], so $u \in R$. Then $\lambda(u)^{-1}W_u|J_{1/2} = \lambda(u)^{-1}u^\rho$ is multiplication by u^{-1}, so $u = 1$. □

Theorem 5.3 implies that $\Gamma^e(Q, I)$ is the subgroup of $\Gamma(Q)$ generated by $a_1 a_2$, $a_i \in J_0'$, $a_1 \equiv a_2 \pmod{IJ}$.

Suppose $\phi \in G$, ϕ fixes J_0, and $\phi(e) = \alpha e$. Let $N(\phi y) = \beta N(y)$ for $y \in J$. For $x \in J_0$,

$$\begin{aligned}\alpha Q(\phi x) &= \alpha N(e + \phi x) = N(\alpha e + \phi x)\\ &= N(\phi(e + x)) = \beta N(e + x) = \beta Q(x).\end{aligned}$$

Then $G_{e,I} = \{\phi \in G_I | \phi \text{ fixes } J_0, \phi(e) = \alpha e, \text{ and } N(\phi y) = \alpha N(y) \text{ for } y \in J\}$ and

$$\begin{aligned}S_{e,I} &= \{\phi \in S_I | \phi \text{ fixes } e \text{ and } J_0\} \qquad \text{(viii)}\\ &= \{\phi \in G_{e,I} | \phi \text{ fixes } e\}.\end{aligned}$$

COROLLARY 5.4. *$u \to W_u$ is an isomorphism of $\mathrm{Spin}(Q, I)$ onto $S_{e,I}$.* □

6. Generating S_I by algebraic transvections. In this section we apply Corollary 5.4 to prove that $S_I = T_I$ for any ideal I of R, so S_I is generated by algebraic transvections.

LEMMA 6.1. *Let X be a 2×2 matrix with entries in a commutative and associative subalgebra A of $\mathfrak{D}$. Assume $X \equiv 1 \pmod{I\mathfrak{D}_2}$. If $I = R$, assume further that some entry of X has norm in $R - m$. Then $(\det X)e_1 + e_2 = HXK$, where H and K are products of elements of the form*

$$(1 + p_s e_{ij})(1 + q_s e_{ji})(1 - p_s e_{ij}),$$

$p_s \in A$, $q_s \in A \cap I\mathfrak{D}$, $i \neq j$.

PROOF. If $y \in A$ and $n(y) \in R - m$, then $y^d = t(y)1 - y$ and $y^{-1} = n(y)^{-1}y^d$ are also elements of A. First assume that $I \neq R$. Let

$$X = (1 + a)e_1 + be_{12} + ce_{21} + (1 + g)e_2,$$

where lower-case letters denote elements of $A \cap I\mathfrak{D}$. Then

$$X(1 + e_{21})\big(1 - (1 + c + g)^{-1}ge_{12}\big)(1 - e_{21}) = (1 + r)e_1 + se_{12} + te_{21} + e_2.$$

Multiplying on the left by $1 - se_{12}$ and on the right by $1 - te_{21}$ gives $(\det X)e_1 + e_2$. Next assume that $I = R$ and some entry of X is invertible. We can make an adjacent entry 1 by adding a multiple of a row or column to another. Repeating this, we can make the e_2 entry 1 and conclude as above. □

PROPOSITION 6.2. $S_{e,I} \subset T_I$.

PROOF. By Corollary 5.4, every element of $S_{e,I}$ has the form

$$\big(\phi_1 U_{e+a_1}\phi_1^{-1}\big) \cdots \big(\phi_n U_{e+a_n}\phi_n^{-1}\big),$$

$\phi_i \in G$, $a_i \in J_0'$, $a_i \equiv f \pmod{IJ_0}$, $\Pi Q(a_i) = 1$. For $\alpha \in R$ and $x \in J_0$,

$$U_{\alpha e+f}U_{e+x} = U_{\alpha e+x} = U_{e+x}U_{\alpha e+f}$$

by the Peirce relations. It follows that we can replace each U_{e+a_i} by $U_{Q(a_i)^{-1}e+a_i}$, since $\Pi Q(a_i) = 1$. Hence it suffices to prove that $U_X \in T_I$ for $X = Q(x)^{-1}e + x$, $x \in J_0'$, $x \equiv f \pmod{IJ_0}$.

Write $x = \alpha_2 e_2 + \alpha_3 e_3 + a[23]$ and let A be the subalgebra of $\mathfrak{D}$ generated by a. We claim that X is a product of matrices of the form

$$(1 + \gamma_j p e_{ij})(1 + \gamma_i q e_{ji})(1 - \gamma_j p e_{ij}),$$

$p \in A$, $q \in A \cap I\mathfrak{D}$. Since $Q(x) \in R - m$, either α_2 or $n(a)$ is a unit, so we can apply Lemma 6.1 to x. Then X can be multiplied by matrices of the required form to give $Q(x)^{-1}e + Q(x)e_2 + e_3$. Applying Lemma 6.1 again establishes the claim.

Let π be the canonical involution of $\mathfrak{D}_3$ taking Y to $\gamma^{-1}Y^{dt}\gamma$, where Y^{dt} is the conjugate transpose of Y. For $Z \in J$, $U_X Z = XZX = X^{\pi}ZX$. Write X as a product of matrices as in the last paragraph. Since these matrices have coefficients in A, they can be associated with Z in any way. Since

$$T_{b[ji],e_j}Z = (1 + \gamma_i b e_{ji})^{\pi}Z(1 + \gamma_i b e_{ji})$$

for $b \in \mathfrak{D}$, it follows that U_X is a product of terms of the form

$$T_{p[ij],e_i}T_{q[ji],e_j}T_{p[ij],e_i}^{-1},$$

$p \in \mathfrak{D}$, $q \in I\mathfrak{D}$, so $U_X \in T_I$. □

LEMMA 6.3. *Let $\phi \in S_I$ fix each Re_i. Then there is $\psi \in T_I$ such that $\psi\phi$ fixes e_1, Re_2, and Re_3.*

PROOF. By Lemma 3.2(1), ϕ fixes each $\mathfrak{O}[jk]$. Let $\phi(e_1) = \alpha e_1$ and $\phi(1[23]) = a[23]$, $\alpha \equiv 1 \pmod I$, $a \equiv 1 \pmod{I\mathfrak{O}}$. Since $\phi \in S$, ϕ preserves $T(x^\#, y) = \partial_y N|_x$. Then

$$-\gamma_2\gamma_3 = T\big(1[23]^\#, e_1\big) = T\big(\phi(1[23])^\#, \phi(e_1)\big)$$
$$= T\big(a[23]^\#, \alpha e_1\big) = -\gamma_2\gamma_3 \alpha n(a)$$

so $\alpha n(a) = 1$ and a is invertible. Define $\psi\colon J \to J$ by $\psi(Z) = Y^\pi Z Y$ for $Y = ae_1 + a^d e_2 + n(a)^{-1}e_3$ and canonical involution π. $\psi \in T_I$ by the proof of Proposition 6.2, and $\psi\phi$ fixes e_1, Re_2, and Re_3. □

THEOREM 6.4. *$S_I = T_I$ for any ideal I of R.*

PROOF. Clearly $T_I \subseteq S_I$. Conversely, let $\phi \in S_I$. By Proposition 2.1 and Lemma 6.3, there is $\psi \in T_I$ such that $\psi\phi$ fixes e_1, Re_2, and Re_3. $\psi\phi$ fixes $J_0(e_1)$ [Lemma 3.2(1)], so (viii) shows that $\psi\phi \in S_{e_1,I}$. We are done by Proposition 6.2. □

Combining Theorem 6.4 and Lemma 3.5 yields:

COROLLARY 6.5. (1) *S_I is generated by $\phi T_{a[12],e_1}\phi^{-1}$, $\phi \in T'$, $a \in I\mathfrak{O}$.*

(2) *$S = T'$.*

(3) *If $I \subset K$ are ideals of R, the natural map from $S_K(J)$ to $S_{K/I}(J/IJ)$ is surjective.* □

COROLLARY 6.6. *S is generated by U_X, where $X \in J$, $N(X) = 1$, and the coefficients of X lie in a subalgebra of $\mathfrak{O}$ generated by a single element.*

PROOF. $U_X \in S$ if $N(X) = 1$ [Corollary 1.5]. Conversely, by Corollary 6.5(2), it suffices to write $T_{p[ji],e_j}$ as a product of such U_X's for $p \in \mathfrak{O}$. First assume that p is invertible. Set

$$X_1 = p^{-1}[ij] - \gamma_i^{-1}\gamma_j^{-1}n(p)e_k \quad \text{and} \quad X_2 = X_1 + \gamma_i\gamma_j e_j.$$

$X_i \in J$, $N(X_i) = 1$, and $(1 + \gamma_i p e_{ji})X_1 = X_2$. For $Z \in J$,

$$T_{p[ji],e_j}Z = (1 + \gamma_i p e_{ji})^\pi Z(1 + \gamma_i p e_{ji}) = \big(X_2X_1^{-1}\big)^\pi Z\big(X_2X_1^{-1}\big)$$
$$= X_1^{-1}X_2 Z X_2 X_1^{-1} = U_{X_1^{-1}}U_{X_2}Z,$$

since X_1 and X_2 have coefficients in the subalgebra of $\mathfrak{O}$ generated by p. Next assume that p is not invertible, so $n(p) \in m$. By considering images in $\mathfrak{O}/m\mathfrak{O}$, we can find $q \in \mathfrak{O}$ such that $n(q)$ and $n(p + q)$ are units, so q and $p + q$ are invertible. Since

$$T_{p[ji],e_j} = T_{p+q[ji],e_j}T_{-q[ji],e_j}$$

[5, p. 49], we are done by the first case. □

7. Subgroups of Γ normalized by S. We combine Corollary 3.6 and Theorem 6.4 to prove the main theorem classifying the subgroups of Γ normalized by S. We apply the theorem to determine the normal subgroups of S, PS, G, and PG.

THEOREM 7.1. *A subgroup N of Γ is normalized by S if and only if $S_I \subseteq N \subseteq (R - m)\Gamma_I$ for an ideal I of R. I is uniquely determined by N.*

PROOF. Let N be a subgroup of Γ normalized by S. $\{p \in \mathfrak{O} | T_{p[12],e_1} \in N\} = I\mathfrak{O}$ for an ideal I of R [Lemma 3.4]. N contains S_I, by Corollary 6.5(1). Let $(\phi, \sigma) \in N$ and $a \in I$. Let $x = \phi^{-1}(1[12])$ and $y = \phi^*(e_1)$; $y \in \Pi$, $T(x, y) = 0$. Since $S_I \subseteq N$, N contains

$$\phi T_{ax,y}\phi^{-1} = T_{a^\sigma[12],e_1}.$$

Then $a^\sigma \in I$ and $I^\sigma \subseteq I$. Since $\phi^{-1} \in N$, $I^\sigma = I$. Let

$$\Gamma^I = \{\phi \in \Gamma | \phi(IJ) = IJ\} = \{(\phi, \sigma) \in \Gamma | I^\sigma = I\}.$$

Γ^I is a subgroup of Γ containing N, G, and Γ_I, and there is a homomorphism f: $\Gamma^I \to \Gamma(J/IJ)$ taking each element of Γ^I to the norm semisimilarity it induces on J/IJ as an R/I-algebra. The kernel of f is Γ_I. We claim that $f(N) \subseteq R/I - m/I$, so $S_I \subseteq N \subseteq (R - m)\Gamma_I$. Suppose not. Since $f(N)$ is normalized by $S(J/IJ)$ [Corollary 6.5(3)], $f(N)$ contains $S_{K/I}(J/IJ)$ for an ideal K of R, $K \supset I$, $K \neq I$ [Corollary 3.6 and Theorem 6.4]. Corollary 6.5(1) and (vi) show that $S_{K/I}(J/IJ)$ is generated by elements $[\tau, \phi]$, $\tau \in S(J/IJ)$, $\phi \in S_{K/I}(J/IJ)$. Since $f(N)$ contains $S_{K/I}(J/IJ)$, $\phi = f(\psi)$ for $\psi \in N$. $\tau = f(\eta)$ for $\eta \in S(J)$ [Corollary 6.5(3)], so $[\tau, \phi] = f([\eta, \psi])$. $[\eta, \psi] \in S \cap N$, since Γ normalizes S and S normalizes N. Thus $f(S \cap N)$ contains $S_{K/I}(J/IJ)$. Since $S \cap N$ contains S_I, the kernel of f restricted to S, it follows that $S \cap N$ contains S_K, a contradiction.

Conversely, assume that $S_I \subseteq N \subseteq (R - m)\Gamma_I$. Let $\phi \in N$, $\tau \in S$. $[\tau, \phi] \in S$, since Γ normalizes S. Define f: $\Gamma^I \to \Gamma(J/IJ)$ as above. Since $N \subseteq (R - m)\Gamma_I$, $f(\phi)$ is a scalar multiplication, so $f([\tau, \phi]) = 1$. Then $[\tau, \phi] \in S_I \subseteq N$ and $\tau\phi\tau^{-1} \in N$, as required.

To show the uniqueness of I, let I and K be ideals of R, $I \not\supseteq K$. Every element of $(R - m)\Gamma_K$ induces scalar multiplication on J/KJ, while, for $a \in I - K$, $T_{a[12],e_1} \in S_I$ does not. Then S_I is not contained in $(R - m)\Gamma_K$, implying uniqueness. □

Combining Lemma 3.3 and Theorem 7.1 gives:

COROLLARY 7.2. *A subgroup N of $P\Gamma$ is normalized by S if and only if $PS_I \subseteq N \subseteq P\Gamma_I$ for some ideal I of R. I is uniquely determined by N.* □

COROLLARY 7.3. *If N is a subgroup of G, the following are equivalent:*

(1) *N is normal.*

(2) *N is normalized by S.*

(3) *$S_I \subseteq N \subseteq (R - m)G_I$ for an (unique) ideal I of R.*

PROOF. Only (3) $\Rightarrow$ (1) remains to be proved. If $S_I \subseteq N \subseteq (R - m)G_I$, for $\phi \in N$ and $\tau \in G$ one verifies that $[\tau, \phi] \in S_I \subseteq N$, so N is normal. □

COROLLARY 7.4. *If N is a subgroup of PG, the following are equivalent:*

(1) *N is normal.*

(2) *N is normalized by PS.*

(3) *$PS_I \subseteq N \subseteq PG_I$ for an (unique) ideal I of R.* □

We now apply Corollaries 7.3 and 7.4 to determine the normal subgroups of S, PS, G, and PG. Theorem 6.4 and Corollary 7.4 give:

COROLLARY 7.5. *The distinct normal subgroups of PS are PT_I, I an ideal of R.* □

When R is field, Corollary 7.5 states that PS is simple, as proved in [5, p. 49].

For $\beta \in R - m$, define $\phi_\beta: J \to J$ by

$$\phi_\beta\Big(\sum \alpha_i e_i + \sum a_i[jk]\Big) = \beta^{-2}\alpha_1 e_1 + \beta^{-2}\alpha_2 e_2 + \beta^4\alpha_3 e_3 + \beta a_1[23] + \beta a_2[31] + \beta^{-2}a_3[12].$$

$\phi_\beta \in S$, by direct verification. Let $SC_I = S \cap (R - m)G_I = \{\phi \in S |\ \phi$ induces scalar multiplication on $J/IJ\}$. If I is an ideal of R and $\beta^3 \equiv 1 \pmod I$, ϕ_β induces multiplication by β on J/IJ and $\phi_\beta \in SC_I$. For $L \subseteq R - m$, let $\phi_L = \{\phi_\beta | \beta \in L\}$. For an ideal I of R, let $D_I = \{\alpha \in R - m | \alpha \equiv 1 \pmod I\}$ and $E_I = \{\alpha \in R - m | \alpha^3 \equiv 1 \pmod 1\}$.

COROLLARY 7.6. *The distinct normal subgroups of S are $T_I\phi_L$, where I is an ideal of R and L is a subgroup of $R - m$ such that $D_I \subseteq L \subseteq E_I$.*

PROOF. By Corollary 7.3, we must show that the subgroups N of S such that $S_I \subseteq N \subseteq SC_I$ have the above form. Each $\tau \in SC_I$ induces scalar multiplication on J/IJ by some $\alpha \in E_I$, where α is determined up to a multiple of D_I. We define a homomorphism $\eta: SC_I \to E_I/D_I$ by $\eta(\tau) = \alpha D_I$. η is onto, since $\eta(\phi_\alpha) = \alpha D_I$ for $\alpha \in E_I$. Since $S_I = T_I$ is the kernel of η, the corollary follows. □

For $\delta \in D_I$, define $\psi_\delta: J \to J$ by

$$\psi_\delta\Big(\sum \alpha_i e_i + \sum a_i[jk]\Big) = \delta\alpha_1 e_1 + \delta\alpha_2 e_2 + \delta^{-1}\alpha_3 e_3 + a_1[23] + a_2[31] + \delta a_3[12].$$

One sees that $N(\psi_\delta x) = \delta N(x)$ for $x \in J$, so $\psi_\delta \in G_I$. For $H \subseteq D_I$, let $\psi_H = \{\psi_\delta | \delta \in H\}$. Define a homomorphism W: $(R - m) \times D_I \to (R - m)G_I$ by $W(\beta, \delta) = \beta\psi_\delta$.

COROLLARY 7.7. *The distinct normal subgroups of G are $T_I W(L)$, where L is a subgroup of $(R - m) \times D_I$ containing (τ, τ^{-3}), $\tau \in D_I$.*

PROOF. If $\phi \in G_I$, $N(\phi x) = \delta N(x)$ for $\delta \in D_I$, so $\psi_\delta^{-1}\phi \in S_I$. Then $G_I = \psi_{D_I} S_I$ and $(R - m)G_I = W((R - m) \times D_I)S_I$. Consider the homomorphism $(\beta, \delta) \to W(\beta, \delta)S_I$ of $(R - m) \times D_I$ onto $(R - m)G_I/S_I$. If (β, δ) belongs to the kernel, $\beta\psi_\delta \in S_I$. Applying $\beta\psi_\delta$ to J/IJ shows that $\beta \in D_I$. Since $N(\beta\psi_\delta x) = \beta^3\delta N(x)$, $\delta = \beta^{-3}$. Thus the kernel is $\{(\tau, \tau^{-3}) | \tau \in D_I\}$, and we are done by Corollary 7.3. □

COROLLARY 7.8. *The distinct normal subgroups of PG are $PT_I\psi_H$, where H is a subgroup of D_I containing τ^3, $\tau \in D_I$.* □

8. Norm semisimilarities and collineations. In this final section we prove that every collineation of two octonion planes is induced by a norm semisimilarity of the underlying algebras. In particular, $P\Gamma$ is the collineation group of PJ.

Let $\mathfrak{D}', J' = H(\mathfrak{D}', \gamma')$, and N' be defined over a local ring (R', m') as their namesakes are defined over R. A norm semisimilarity $\phi: J \to J'$ induces a collineation $P\phi: PJ \to PJ'$ as in §2. We define a four-point to be an ordered quadruple $(a_{1*}, a_{2*}, a_{3*}, a_{4*})$ of points of PJ such that $a_{i*} \nsim (a_j \times a_k)^*$ for i, j, k distinct.

LEMMA 8.1. *PG is transitive on four-points.*

PROOF. By Proposition 2.1(3), it suffices to prove that if $(e_{1*}, e_{2*}, e_{3*}, b_*)$ is a four-point and $c = \sum \gamma_i e_i + \sum 1[jk]$ then there is $\phi \in \Gamma$ such that $P\phi$ fixes each e_{i*} and $P\phi(b_*) = c_*$. Let $b = \sum \gamma_i \beta_i e_i + \sum b_i[jk]$. Since $b_* \nsim (e_j \times e_k)^* = e_i^*$, each $\beta_i \in R - m$; then, since $b \in \Pi$, each $n(b_i) \in R - m$ and b_i is invertible. For $X \in \mathfrak{D}_3$, define $\phi_X: J \to J$ by $\phi_X(Z) = XZX^\pi$, π as in the proof of Proposition 6.2. If $X = b_1 e_1 + b_1^{-1} e_2 + e_3$, $\phi_X \in S$, by the proof of Proposition 6.2. $P\phi_X$ fixes each e_{i*}, so we can replace b by $\phi_X b$ and assume that $b_1 = 1$. Define $\tau: J \to J$ by

$$\tau\Big(\sum \alpha_i e_i + \sum a_i[jk]\Big) = \alpha_1 e_1 + n(b_2)\alpha_2 e_2 + n(b_2)^{-1}\alpha_3 e_3 + b_2^d a_1 b_2^{-1d}[23] + b_2^{-1} a_2[31] + a_3 b_2[12].$$

$\tau \in S$, by direct verification using the Moufang identities and the relations $t(xy) = t(yx)$ and $t([x, y, z]) = 0$ [7, pp. 16, 163]. We can replace b by τb and assume that $b_1 = 1 = b_2$. Take $\psi_{\beta_3} \in G$ as in Corollary 7.7. Replacing b by $\psi_{\beta_3} b$ makes $\beta_3 = 1$ and $b_1 = 1 = b_2$, so $b = c$. □

The next lemma can be proved exactly as in [**5**, p. 36].

LEMMA 8.2. (1) *$a_* \sim x^*$ if and only if there is $c_* | x^*$ such that $a_* \sim c_*$.*
(2) *If $a_* \sim x^*$ and $c_* | x^*$, then either $a_* \sim c_*$ or $x^* \sim (a \times c)^*$.* □

LEMMA 8.3. *Let $(a_{1*}, a_{2*}, a_{3*}, a_{4*})$ be a four-point and let W be a collineation of PJ that fixes the a_{i*} and all points on $(a_1 \times a_2)^*$ not connected to a_{1*}. Then W is the identity.*

PROOF. Let $a_5 = (a_1 \times a_3) \times (a_2 \times a_4)$. We repeatedly apply Lemma 8.2 and its dual. $(a_1 \times a_3)^* \nsim (a_2 \times a_4)^*$, else $a_{1*} \sim (a_2 \times a_4)^*$; so $a_5 \in \Pi$. $a_{5*} \nsim a_{1*}$, else $a_{1*} \sim (a_2 \times a_4)^*$. Then $a_{5*} \nsim (a_1 \times a_2)^*$, else $(a_1 \times a_2)^* \sim (a_1 \times a_5)^* = (a_1 \times a_3)^*$ and $a_{2*} \sim (a_1 \times a_3)^*$. This implies that a_{5*} is not connected to any point on $(a_1 \times a_2)^*$ or $(a_3 \times a_4)^*$ (by symmetry).

CLAIM 1. W fixes all $c_* | (a_3 \times a_4)^*$ such that $c_* \nsim a_{3*}$. Let $f = [(c \times a_5) \times (a_1 \times a_2)]$. $f \in \Pi$, since a_{5*} is not connected to any points on $(a_1 \times a_2)^*$ or $(a_3 \times a_4)^*$. $c_* = [(f \times a_5) \times (a_3 \times a_4)]_*$. If we show that $f_* \nsim a_{1*}$, then W fixes f_* and hence c_*, as required. Assume that $f_* \sim a_{1*}$. Then $a_{1*} \sim (f \times a_5)^* = (c \times a_5)^*$, so $(c \times a_5)^* \sim (a_1 \times a_5)^* = (a_1 \times a_3)^*$ and $c_* \sim (a_1 \times a_3)^*$. Since $c_* \nsim a_{3*}$, $(a_1 \times a_3)^* \sim (c \times a_3)^* = (a_3 \times a_4)^*$, so $a_{1*} \sim (a_3 \times a_4)^*$, a contradiction.

CLAIM 2. W fixes all points f_* on $(a_1 \times a_2)^*$ or $(a_3 \times a_4)^*$. By symmetry, Claim 1 shows that W fixes all points on $(a_3 \times a_4)^*$ not connected to both a_{3*} and a_{4*} and all points on $(a_1 \times a_2)^*$ not connected to both a_{1*} and a_{2*}. Thus we can assume that $f_* | (a_3 \times a_4)^*$, $f_* \sim a_{3*}$, and $f_* \sim a_{4*}$. Let $a_{6*} = [(a_1 \times a_2) \times (a_3 \times a_4)]_*$. $a_{6*} \nsim a_{3*}$, so we can assume that $a_{6*} = e_{1*}$ and $a_{3*} = e_{2*}$ [Proposition 2.1(1)]. Since

$e_3^* = (a_3 \times a_6)^* = (a_3 \times a_4)^*$, $f = \delta_1 e_1 + p[12] + \delta_2 e_2$. $\delta_1 \in m$, since $f_* \sim a_{3*}$. Let $g = e_1 + q[12] + \gamma_1\gamma_2 n(q)e_2$ for $q \in \mathfrak{O}$ to be chosen.

$$g \times f \equiv (\delta_2 - \gamma_1\gamma_2 n(p, q))e_3 \pmod{mJ}.$$

Since either $p \in J - mJ$ or $\delta_2 \in R - m$, we can choose q so that $g \times f \in J - mJ$ and $n(q)$ is a unit. Then $g_* | (a_3 \times a_4)^*$ and g is not connected to either a_{3*}, a_{6*}, or f_*. a_{1*} and a_{2*} are not connected to $(a_3 \times g)^* = (a_3 \times a_4)^*$, and g_* is not connected to $(a_1 \times a_2)^*$ (else $g_* \nsim a_{6*}$ implies that $(a_1 \times a_2)^* \sim (g \times a_6)^* = (a_3 \times a_4)^*$). Then $(a_{1*}, a_{2*}, a_{3*}, g_*)$ is a four-point, by the symmetry of $b_{i*} \nsim (b_j \times b_k)^*$. W fixes g_*, by Claim 1. Applying Claim 1 again shows that W fixes all points on $(a_3 \times g)^*$ not connected to both a_{3*} and g_*, so W fixes f_*.

CLAIM 3. W fixes all $f_* \nsim (a_3 \times a_4)^*$. Let $g = (f \times a_3) \times (a_1 \times a_2)$, $g \in \Pi$. Since $g_* | (a_1 \times a_2)^*$, W fixes g_* and $g_* \nsim a_{3*}$. Then W fixes $(g \times a_3)^* = (f \times a_3)^*$. By symmetry, W fixes $(f \times a_4)^*$. Moreover, $(f \times a_4)^* \nsim (f \times a_3)^*$, else $a_{4*} \sim (f \times a_3)^*$ would contradict $f_* \nsim (a_3 \times a_4)^*$. Thus W fixes

$$[(f \times a_3) \times (f \times a_4)]_* = f_*.$$

CLAIM 4. W fixes all points on lines $x^* \nsim (a_3 \times a_4)^*$. Since we can assume that $x^* = e_1^*$ and $(a_3 \times a_4)^* = e_2^*$, we can find points b_{1*} and b_{2*} on x^* and c_{1*} and c_{2*} on $(a_3 \times a_4)^*$ such that $(b_{1*}, b_{2*}, c_{1*}, c_{2*})$ is a four-point. By Claim 3, W fixes b_{1*} and b_{2*}. Then W fixes all points on $(b_1 \times b_2)^* = x^*$, by Claim 2.

We now prove the lemma. By Claim 4, W fixes all points on $(a_1 \times a_3)^*$. Applying Claim 4 again shows that W fixes all points on lines not connected to at least one of $(a_1 \times a_3)^*$, $(a_3 \times a_4)^*$, or $(a_1 \times a_2)^*$. Since

$$(a_1 \times a_2)^* \nsim a_{3*} = [(a_1 \times a_3) \times (a_3 \times a_4)]_*,$$

we can assume that $(a_1 \times a_3)^* = e_1^*$, $(a_3 \times a_4)^* = e_2^*$, and $(a_1 \times a_2)^* = e_3^*$. Examination shows that no line is connected to all three of these lines, so W is the identity. □

THEOREM 8.4. *Any collineation W: $PJ \to PJ'$ has the form $P\phi$ for a norm semisimilarity ϕ: $J \to J'$. In particular, $P\Gamma$ is the collineation group of PJ.*

PROOF. One sees directly that $X \to X\gamma$ is a norm similarity of $H(\mathfrak{O}_3, 1)$ and $H(\mathfrak{O}_3, \gamma)$, so we can assume that $\gamma = 1$ and $\gamma' = 1'$. By Lemma 8.1, we can assume that $W(e_{i*}) = e'_{i*}$ and $W(c_*) = c'_*$ for $c = \Sigma e_i + \Sigma 1[jk]$. Then as in [5, p. 40] there is a ring isomorphism τ: $\mathfrak{O} \to \mathfrak{O}'$ defined by

$$W(e_2 + a[23] + n(a)e_3)_* = (e_2 + a^\tau[23] + n'(a^\tau)e_3)_*,$$

$a \in \mathfrak{O}$. $\tau(R) = R'$ [Lemma 1.11], so $t'(\tau x) = \tau(t(x))$ follows from the relation $x_i^2 - t(x_i)x_i + n(x_i)1 = 0$ for a basis $\{1, x_i\}$ of $\mathfrak{O}$. Then $\tau(x^d) = (\tau x)^{d'}$, so $n'(\tau x) = \tau(n(x))$. Thus applying τ to each coordinate defines a semilinear algebra isomorphism ϕ: $H(\mathfrak{O}_3, 1) \to H(\mathfrak{O}'_3, 1')$. Then W and $P\phi$ agree on e_{i*}, c_*, and all points on e_1^* not connected to e_{3*}, so $P\phi = W$ [Lemma 8.3]. □

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Department of Mathematics, The University of Michigan-Flint, Flint, Michigan 48503